

# H-Dyons and S-Duality

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## Abstract

We present a relatively simple argument showing that the H-dyon states required by S-duality of the heterotic string on  $T^6$  are present provided that the BPS dyons required by S-duality of  $N = 4$  supersymmetric Yang-Mills theory are present. We also conjecture and provide evidence that H-dyons at singularities where the nonperturbative gauge symmetry is completely broken are actually BPS dyons.

## 1. Introduction

The known superstring theories have all been conjectured to be related to each other through dualities acting on the coupling constant and the target space [1,2]. In other words, spacetime dualities are in some sense equivalent to worldsheet dualities. Evidence has mounted during the last several years that these conjectures are correct. The first conjectured weak-strong coupling duality of superstrings involved the heterotic string compactified on a six-torus. This theory is believed to have an  $SL(2, \mathbf{Z})$  symmetry (S-duality) acting on the complex coupling as well as the electric and magnetic excitations. Supporting evidence and predictions of heterotic string S-duality were presented in Ref. [3]. One of these predictions concerns the existence of the so called “H-dyon” states. The purpose of this note is to argue that these states exist and to reveal their whereabouts.

If S-duality is truly a symmetry of the heterotic string on a six-torus, all measurable quantities must be invariant under  $SL(2, \mathbf{Z})$  transformations. These transformations convert elementary electrically charged states into states with both electric and magnetic charges (dyons) at a new value of the coupling. In some cases these dyon states cannot decay into any lower energy states that conserve their charges and are expected to be stable states. S-duality predicts that their degeneracies, in these cases, should be equal to those of the corresponding elementary states. The H-dyon states to be discussed here are the S-duality transforms of a certain class of elementary string states to be specified in section two. In that section we will refresh the reader’s memory about some facts related to S-duality. Under the special weak-strong coupling transformation of  $SL(2, \mathbf{Z})$ , the above class of elementary string states become the H-monopoles. The evidence that H-monopole degeneracies agree with the predictions of S-duality is discussed in section three. In section four we discover the H-dyons according to the prediction so long as another class of dyons, the BPS dyons, is also detected. We summarize the results in section five. Along the way we conjecture that H-monopoles or H-dyons at singularities where the nonperturbative gauge symmetry is broken completely but the gauge group is abelian are actually BPS monopoles or BPS dyons.

## 2. Review of S-Duality

To set the stage for our argument let us review the background details about S-duality of the heterotic string. The discussion will follow Ref. [3]. Following the convention that the right moving sector is supersymmetric whereas the gauge symmetry resides in the

left moving sector, the masses of perturbative string states can be written in the Neveu-Schwarz sector by the following relation for a particular choice of background values on the six-torus of the metric, antisymmetric tensor, and sixteen  $U(1)$  gauge fields from the ten dimensional gauge group  $SO(32)$  or  $E_8 \times E_8$ :

$$M^2 \propto \frac{1}{(Im\tau)_{\alpha'}}(\vec{p}_R^2 + 2N_R - 1) = \frac{1}{(Im\tau)_{\alpha'}}(\vec{p}_L^2 + 2N_L - 2). \quad (2.1)$$

The Ramond sector masses are degenerate by supersymmetry. The left and right internal momenta and winding vectors  $(\vec{p}_L, \vec{p}_R) \in \Lambda_{22,6}$  where  $\Lambda_{22,6}$  is an even, self-dual Lorentzian lattice also determined by the position in the moduli space. The  $N_L, N_R$  are left and right moving oscillator numbers,  $-1/2$  and  $-1$  are the right and left moving vacuum energies, and  $\tau$  is the asymptotic value of the complex coupling ( $\tau = \frac{\theta}{2\pi} + \frac{i}{g^2}$ ;  $\theta$  is the axion,  $g^2$  is the string loop expansion parameter). The string tension is  $T = 1/2\pi\alpha'$ .

It is known that the  $N = 4$  supersymmetry of the heterotic string on a six-torus protects the masses of states that satisfy a Bogomol'nyi bound from receiving quantum corrections. The elementary electrically charged string states breaking half of the supersymmetry and saturating this bound have  $N_R = 1/2$  but are otherwise arbitrary. These states satisfy the relation:  $N_L - 1 = \frac{1}{2}(\vec{p}_R^2 - \vec{p}_L^2)$ . There are also states not seen in the perturbative string theory that contain both electric and magnetic charges. One can write these charges in the form

$$(\vec{Q}_{el}, \vec{Q}_{mag}) = (\frac{1}{Im\tau}(\vec{p} + Re\tau\vec{q}), \vec{L}q) \quad (2.2)$$

where  $(\vec{p}, \vec{q}) \in \Lambda_{22,6}$ ,  $L$  is the  $28 \times 28$  matrix with

$$L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix}, \quad (2.3)$$

and  $I_n$  is the  $n \times n$  identity matrix. One can also write the left and right moving vectors as projections onto the subspaces  $L = \mp 1$ :

$$\begin{aligned} p_L^a &= \frac{1}{2}(I_{28} - L)_{ab}p^b \\ p_R^a &= \frac{1}{2}(I_{28} + L)_{ab}p^b \end{aligned} \quad (2.4)$$

The first six components of  $Q_{el}$  are charges with respect to the gauge fields from the dimensional reduction of the ten dimensional metric on  $T^6$ , the next six are charges with

respect to the gauge fields from the antisymmetric tensor, and the last sixteen are charges of the sixteen  $U(1)$  fields. Under the  $SL(2, \mathbf{Z})$  transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the coupling and lattice vectors transform as follows:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad (2.5)$$

$$\begin{pmatrix} \vec{p} \\ \vec{q} \end{pmatrix} \rightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \vec{p} \\ \vec{q} \end{pmatrix}.$$

The weak-strong coupling transformation corresponds to the  $SL(2, \mathbf{Z})$  matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The mass formula can be written in a manifestly  $SL(2, \mathbf{Z})$  invariant fashion. One would like to show that the spectrum of charges is also invariant under  $SL(2, \mathbf{Z})$ . One must, therefore, determine the degeneracies of various dyonic states. In general, an elementary string state  $\vec{p} = \vec{e}$  where  $\vec{e}$  is an elementary lattice vector transforms into a state  $m\vec{e} + n\vec{\tilde{e}}$  where  $\vec{\tilde{e}}$  is a dual lattice vector (also a lattice vector in this case). In the case that  $(m, n)$  are relatively prime, the mass formula implies that the corresponding state should be stable, and  $SL(2, \mathbf{Z})$  invariance implies that the degeneracy should equal the corresponding degeneracy of elementary states.

The sixteen states (eight Neveu-Schwarz states and eight Ramond states from the right moving sector) with  $\vec{p}_L^2 - \vec{p}_R^2 = 2$  ( $N_L = 0$ ) which are electrically charged under one of the sixteen  $U(1)$ 's have an interpretation as elementary charged particles arising from an  $N = 4$  supersymmetric Yang-Mills theory in four dimensions in which the gauge group is broken to an abelian subgroup by expectation values of the scalar fields. Since the masses of these particles can vanish in the field theory limit, gravity can be ignored. S-duality changes these states into BPS dyons. It was shown in Ref. [4] that for the  $SU(2)$  case, the theory of the sixteen magnetically charged states resulting from a weak-strong coupling  $SL(2, \mathbf{Z})$  transformation at low energies and weak coupling is an  $N = 4$  supersymmetric quantum mechanics on the moduli space of classical monopole solutions as expected by S-duality. It was conjectured [5] that there should be unique harmonic forms on this moduli

space for each  $(n, m)$  with  $n, m$  relatively prime integers and  $n$  the number of magnetic charges and  $m$  an integer such that the electric charge is proportional to  $m + \frac{n\theta}{2\pi}$ .

There are  $24 \times 16$  states with  $\vec{p}_L^2 - \vec{p}_R^2 = 0$  since there are 24 left moving oscillators with  $N_L = 1$  (8 space-time oscillators and 16  $U(1)$  oscillators). These states arise as momentum or winding states on the six-torus. They are electrically charged with respect to a gauge field coming from the metric or antisymmetric tensor. It was conjectured [3] that there should be  $16 \times 24$  H-dyon bound states satisfying the Bogomol'nyi bound for each  $(n, m)$  with  $n, m$  relatively prime integers with  $n$  the instanton number and  $m$  the momentum on  $S^1$ . Under the weak-strong coupling transformation the momentum states on an  $S^1$  turn into magnetically charged states with respect to the gauge field from the antisymmetric tensor. They satisfy  $dH = F \wedge F$  corresponding to one instanton lying in  $R^3 \times S^1$ , and they are known as H-monopoles (H is the antisymmetric tensor field strength, F is the gauge field strength). The solitonic states with  $N_L > 1$  are always heavy in the field theory limit and can never be constructed from massless string fields as have been, for example, certain dyons with  $N_L = 1$  at special values of the moduli [6]. To reiterate, our goal is to locate the  $16 \times 24$   $(n, m)$  H-dyon bound states when  $n$  and  $m$  are relatively prime integers.

### 3. H-Monopoles Unmasked

The expected H-monopoles corresponding to  $n = 1$  and  $m = 0$  were shown to exist in Refs. [7,8]. (See also Ref. [9].) The  $E_8 \times E_8$  heterotic string on  $T^6$  is indistinguishable from the  $SO(32)$  one because of T-duality so we may as well consider the H-monopole as an instanton of  $SO(32)$ . Generically, the  $SO(32)$  gauge symmetry is broken to  $U(1)^{16}$  by Wilson lines on  $T^6$ , and the instanton shrinks to zero size [7]. Zero size instantons can occur even when the gauge symmetry is nonabelian. The moduli space of the relevant small instantons was described [7] using heterotic-type I duality with the small instanton being dual to a Dirichlet fivebrane wrapped around the five-torus transverse to  $R^3 \times S^1$ . The D5 brane of the type I theory carries an  $Sp(1)$  vector, a neutral hypermultiplet, and a  $(2, 32)$  half hypermultiplet of  $Sp(1) \times SO(32)$  at maximal symmetry points in the moduli space. Symmetry breaking patterns can be determined by adding Wilson lines of  $Sp(1)$  to  $T^5$  and  $SO(32)$  to  $T^6$ , alternatively by giving expectation values to the charged matter, or by some combination of the two. From the point of view of the heterotic string, the  $Sp(1)$  symmetry is nonperturbative, and the instanton classically remains small so long as this symmetry is not completely broken. Finite size instantons have no enhanced gauge

symmetry. The symmetry can be completely broken at singularities in the moduli space of Wilson lines where the interaction of  $SO(32)$  and  $Sp(1)$  Wilson lines cause a charged hypermultiplet to become massless. Yet in this case, assuming the perturbative gauge symmetry is  $U(1)^{16}$ , the instanton must classically remain small. I would like to conjecture that in the quantum theory a small instanton at such a singularity is equivalent to a BPS monopole (certainly the two moduli spaces are in agreement and the nonperturbative gauge symmetry is broken).

Experience tells us that if the singularity at the location of the small instanton were not smoothed, we should expect extra degrees of freedom to enter there. Heuristically, one can understand that at the singularity a  $U(1)$  of the  $SO(32)$  becomes identified with the  $U(1)$  of the broken  $Sp(1)$ , and one can identify the six Wilson lines in this  $U(1)$  with the six neutral scalars of the  $N = 4$  Yang-Mills theory in four dimensions with gauge group  $SU(2)$ . The monopole is an  $Sp(1)$  monopole that breaks half the supersymmetry on the brane ( $N = 4$  in four dimensions is broken to  $N = 2$  at the singularity). If we T-dualize on the  $p$  circles with  $Sp(1)$  Wilson lines, we obtain a  $5 - p$  brane that touches a  $9 - p$  brane at the singularity so the dependence on separations in the  $p$  dimensions has disappeared at this point as expected for a monopole solution.

In Ref. [7] the moduli space of H-monopoles was considered at points where the  $SO(32)$  was broken to  $U(1)^{16}$  by Wilson lines and the  $Sp(1)$  was broken to  $U(1)$  in the same way. At generic points the theory was noninteracting modulo the Weyl group, and sixteen states came from quantizing the zero modes of the hypermultiplet. The vector only gave eight states because it was necessary to divide by the Weyl symmetry of  $Sp(1)$  ( $\mathbf{Z}_2$ ). The total degeneracy was, therefore,  $16 \times 8$  states.

The remaining  $16 \times 16$  states were conjectured [7] to come from the sixteen singularities where the two types of Wilson lines interacted such that a charged hypermultiplet became massless. That a normalizable supersymmetric state comes from each singularity was proved by Ref. [8] (see also Ref. [9]) by reducing the gauge theory near the singularity to a quantum mechanics and computing the index [10] of ground states weighted by  $(-1)^F$  where  $F$  is the fermion number. If our conjecture is correct, this result follows from the existence of the BPS monopole.

## 4. H-Dyons Revealed

We would like to prove that the expected H-dyon states are present for  $n, m$  relatively prime and  $m \neq 0$ . Part of our argument will rely on the existence of the corresponding BPS dyon bound state. Again we will have two contributions to the H-dyon states from generic points in the moduli space and from singularities. Let us summarize our strategy. We will first discuss the type I formulation of the generic H-dyons. There we will see that the number of center of mass states is  $16 \times 8$ . In order to determine whether a bound state exists, we will find it easiest to T-dualize to a type I' system and then dimensionally reduce the system to two dimensions where the supersymmetry will be  $N = 8$ . At this point we will imitate the arguments of Ref. [11] to show that there are  $16 \times 8$  generic bound states. We will then argue as in Ref. [7] that the remaining H-dyon states come from sixteen singularities where the supersymmetry is halved. Using arguments from Refs. [11,12,13], we will show that at these singularities the system of H-dyons can be considered, for the purposes of determining the degeneracy of supersymmetric bound states, as an  $N = 4$  supersymmetric quantum mechanics on the moduli space of BPS dyons.

### 4.1. Generic H-Dyons

We begin with the type I formulation of the H-dyons. Let us assume again that  $SO(32)$  is broken to  $U(1)^{16}$ . The maximal gauge symmetry on  $n$  coincident fivebranes is  $Sp(n)$  with matter in a reducible antisymmetric tensor and 16 fundamentals. In addition to  $SO(32)$  Wilson lines, we want to include a special  $Sp(n)$  Wilson line that takes values in one  $U(1)$  breaking the gauge symmetry to  $SU(n) \times U(1)$  with an adjoint hypermultiplet. The fundamentals and antisymmetric tensors of  $SU(n)$  are charged under the Wilson line  $U(1)$  so only the adjoint remains massless. Such a Wilson line  $W^\rho$  in the fundamental  $2n$  representation can take the form

$$W^\rho = A^\rho \otimes I_n \tag{4.1}$$

with

$$A^\rho(\beta^\rho) = \begin{pmatrix} \cos \beta^\rho & -\sin \beta^\rho \\ \sin \beta^\rho & \cos \beta^\rho \end{pmatrix} \tag{4.2}$$

where  $A^\rho(\frac{\pi}{2}) \otimes I_n$  is preserved by  $Sp(n)$  and  $\rho$  indexes one of the five  $S^1$ 's. Notice that the symmetry is broken to  $SU(n) \times U(1)$  for any  $\beta^\rho \neq 0, \pi$  and that  $Sp(k)$  symmetry ( $1 \leq k \leq n$ ) is only achieved on the boundary of the moduli space. There is a subtlety in this problem related to the fact that the Weyl group of  $Sp(n)$  is  $\mathbf{Z}_2^n \times S_n$  rather

than  $S_n$  which is the Weyl group of  $SU(n)$  ( $S_n$  is the symmetric group on  $n$  elements). In general, we can further break the gauge symmetry to  $U(1)^n$  by a Wilson line of the form  $A^\rho(\beta_1^\rho) \oplus A^\rho(\beta_2^\rho) \oplus \dots \oplus A^\rho(\beta_n^\rho)$ . The center of mass vector will be located at  $\beta^\rho = \frac{1}{n}(\beta_1^\rho + \dots + \beta_n^\rho)$  which is invariant under  $S_n$ . Only an overall  $\mathbf{Z}_2$  commutes with this  $S_n$ . A general Weyl transformation will take this center of mass vector multiplet into the center of mass multiplet in another fundamental region of  $\mathbf{Z}_2^{n-1}$ . Thus, we can restrict ourselves to the fundamental region of  $\mathbf{Z}_2^{n-1}$  with  $0 \leq \beta_j^\rho \leq \pi$  or  $-\pi \leq \beta_j^\rho \leq 0$ . In a fundamental region, we can consider the symmetry breaking as follows:

$$Sp(n) \rightarrow SU(n) \times U(1)$$

$$n(2n+1) \rightarrow n(1) + \bar{n}(-1) + \frac{n(n-1)}{2}(1) + \frac{\bar{n}(\bar{n}-1)}{2}(-1) + (n\bar{n}-1)(0) + 1(0) \quad (4.3)$$

This system has  $N = 2$  supersymmetry in six dimensions, away from singularities where charged hypermultiplets become massless or  $Sp(k)$  symmetry ( $1 \leq k \leq n$ ) is restored such that the supersymmetry is reduced to  $N = 1$ . Next we T-dualize on the  $S^1$  transverse to the fivebrane and dimensionally reduce the resulting type I' theory to the  $S^1/\mathbf{Z}_2$  plus time. We will show that the number of center of mass degrees of freedom of the ground states has not changed by the T-duality or dimensional reduction, and it is easiest to count in six dimensions (on the fivebrane) before T-dualizing. The center of mass hypermultiplet is invariant under the Weyl group, and the quantization of fermion zero modes provides sixteen states. The center of mass vector multiplet provides eight states after dividing by  $\mathbf{Z}_2$ . The total number of center of mass states is  $16 \times 8$ .

There are two equivalent pictures of the H-dyons. We can consider them as fivebranes that carry momentum on  $S^1$  or we can consider them as fivebranes interacting with elementary charged states of the string. The T-duality will convert the type I theory on  $T^5 \times S^1$  into a type I' theory on  $T^5 \times S^1/\mathbf{Z}_2$ . In the second picture the  $n$  fivebranes become  $2n$  sixbranes wrapped on  $S^1/\mathbf{Z}_2$ , and the elementary charged states of momentum  $m$  become elementary winding states of winding number  $m$ . At the two orientifold planes the gauge theory is the same  $Sp(n)$  gauge theory that we have discussed in the type I picture. Away from these planes the theory is locally a type IIA theory with  $U(2n)$  gauge symmetry [14]. If we restrict ourselves to  $0 \leq \theta \leq \pi$  where  $\theta$  is the angle on  $S^1$ , the winding can be considered equivalent to putting the tensor product of  $m$  quarks in the fundamental  $\mathbf{2n}$  representation of  $U(2n)$  at  $\theta = 0$  and the tensor product of  $m$  quarks in the fundamental  $\overline{\mathbf{2n}}$  of  $U(2n)$  at  $\theta = \pi$  [11]. The  $\mathbf{Z}_2$  action takes  $\theta$  to  $-\theta$  and the  $\mathbf{2n}$  to the



$\overline{2n}$ . There is, thus, an electric flux between the two orientifold planes which vanishes at the two planes. Thus, we can reduce the problem to counting the number of supersymmetric ground states of the gauge theory on the brane with an electric flux around the  $S^1/\mathbf{Z}_2$ . Since there are no momentum or winding states on the five circles transverse to the  $S^1/\mathbf{Z}_2$  and the problem here is to count the ground states, we can dimensionally reduce the gauge theory on the brane to the  $S^1/\mathbf{Z}_2$  plus time. The  $N = 2$  supersymmetry in six dimensions reduces to  $N = 8$  in two dimensions away from the above mentioned singularities.

Now let us try to understand in detail the gauge theory that we must analyze. We will choose a representation of  $U(2n)$  on the covering space of the orbifold with  $4n$  sixbranes such that an element of the Lie algebra of  $U(2n)$  is written as follows:

$$U = \begin{pmatrix} S_{2n} & A_{2n} \\ A_{2n} & S_{2n} \end{pmatrix}. \quad (4.4)$$

The  $2n \times 2n$  dimensional matrices  $S_{2n}$  and  $A_{2n}$  can be decomposed as follows:

$$\begin{aligned} S_{2n} &= \begin{pmatrix} A_n + iS_n' & S_n' + iS_n'' \\ -S_n' + iS_n'' & A_n - iS_n' \end{pmatrix} \\ A_{2n} &= \begin{pmatrix} A_n' + iS_n^3 & A_n'' + iA_n^3 \\ A_n - iA_n^3 & -A_n' + iS_n^3 \end{pmatrix} \end{aligned} \quad (4.5)$$

where  $A_n$ ,  $A_n'$ ,  $A_n''$ , and  $A_n^3$  are  $n \times n$  dimensional antisymmetric matrices; and  $S_n$ ,  $S_n'$ ,  $S_n''$ , and  $S_n^3$  are  $n \times n$  dimensional symmetric matrices. The gauge group  $Sp(n)$  is generated by  $ReS_{2n}$  and  $ImS_{2n}$ . This  $Sp(n)$  preserves the metric

$$G_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \quad (4.6)$$

The  $\mathbf{Z}_2$  monodromy is embedded in  $U(2n)$  by the matrix

$$M = \begin{pmatrix} I_{2n} & 0 \\ 0 & -I_{2n} \end{pmatrix}. \quad (4.7)$$

This monodromy has the following action:

$$\begin{aligned} S_{2n} &\rightarrow S_{2n} \\ A_{2n} &\rightarrow -A_{2n} \end{aligned}. \quad (4.8)$$

At the orientifold planes,  $A_{2n}$  is projected out, and the remaining gauge symmetry is  $Sp(n)$ . The  $2n$   $U(1)$ 's of  $U(2n)$  will be the following ( $1 \leq i \leq n$ ) :

$$\begin{aligned} G_{\beta_i} &= I_2 \otimes g_{\beta_i} \\ G_{\alpha_i} &= \begin{pmatrix} 0 & g_{\alpha_i} \\ g_{\alpha_i} & 0 \end{pmatrix} \end{aligned} \quad (4.9)$$

where  $g_{\beta_i}$  and  $g_{\alpha_i}$  are  $2n \times 2n$  dimensional matrices such that

$$\begin{aligned}(g_{\beta_i})_{jk} &= \delta_{j,i}\delta_{k,n+i} - \delta_{j,n+i}\delta_{k,i} \\ (g_{\alpha_i})_{jk} &= i\delta_{j,i}\delta_{k,i} + i\delta_{j,n+i}\delta_{k,n+i}\end{aligned}\tag{4.10}$$

The  $G_{\beta_i}$ 's lie in an  $Sp(n)$  subgroup.

If we denote the Wilson lines, parametrized by angles  $\alpha_i$  and  $\beta_i$ , corresponding to the above  $U(1)$ 's as  $W_{\alpha_i}$  and  $W_{\beta_i}$ , the consistency conditions for Wilson lines  $W$  [15]:

$$MWM^{-1} = W^{-1}\tag{4.11}$$

on  $S^1/\mathbf{Z}_2$  and

$$MWM^{-1} = W\tag{4.12}$$

on  $T^5$  imply that only  $W_{\beta_i}$  can exist on  $T^5$ ; whereas, only  $W_{\alpha_i}$  Wilson lines can exist on  $S^1/\mathbf{Z}_2$ . This makes sense since in the type I picture there is nothing on  $T^5$  corresponding to a  $W_{\alpha_i}$  Wilson line, while  $Sp(n)$  ( $W_{\beta_i}$ ) Wilson lines are not allowed on the transverse  $S^1$ . Adding  $W_{\alpha_i}$  Wilson lines to the  $S^1/\mathbf{Z}_2$  corresponds in the type I picture to moving the fivebranes away from  $\theta = 0$ .

In two dimensions we will have eight adjoint scalar multiplets of  $U(2n)$  consisting of an adjoint scalar and two adjoint fermions. These eight multiplets come from the transverse spatial directions to the  $S^1/\mathbf{Z}_2$ . We can separate out the center of mass  $U(1)$  and count the number of center of mass states. This center of mass which commutes with the remaining  $SU(2n)$  is

$$G_\alpha = \frac{1}{n} \sum_{i=1}^n G_{\alpha_i}.\tag{4.13}$$

The monodromy acts as an overall  $\mathbf{Z}_2$  on the scalar multiplets coming from the  $T^5$  but leaves the scalar multiplets corresponding to  $R^3$  invariant. Thus, the number of center of mass states is  $(2^5/2) \times 2^3 = 16 \times 8$ , exactly what we obtained in the T-dual picture of type I. There is another  $U(1) \in Sp(n)$  that has been discussed previously, that is,  $G_\beta = \frac{1}{n} \sum_{i=1}^n G_{\beta_i}$ . Giving scalars that correspond to  $T^5$  Wilson lines  $W_\beta^\rho$  generated by  $G_\beta$  expectation values breaks the  $SU(2n)$  to  $U(1) \times SU(n) \times SU(n)$ . The monodromy exchanges and complex conjugates the two  $SU(n)$ 's, and the  $SU(n)$  subgroup of  $Sp(n)$  is generated by the difference of corresponding pairs of  $SU(n)$  generators.

The next step is to see whether the left over  $SU(2n)$  theory has a bound state. As argued in Ref. [11], the  $N = 8$  supersymmetry in two dimensions allows for a supersymmetric vacuum with a mass gap. The  $m$  wound onebrane should be considered to

contribute a  $SU(2n)$  electric flux away from the fixed points of the the  $S^1/\mathbf{Z}_2$  equivalent to the antisymmetric tensor product of  $m$   $SU(2n)$  fundamentals and a  $U(1)$  flux of charge  $m$ . We can now repeat the analysis of Ref. [11]. The potential energy in the two dimensional gauge theory is

$$V = \text{const} \sum_{i,j=2}^9 \text{Tr}[X^i, X^j]^2 \quad (4.14)$$

where the  $X^i$  are in the adjoint representation of  $SU(n)$ . A supersymmetric quantum vacuum with  $V = 0$  and no charge on the boundaries requires that  $SU(n)$  be broken to  $U(1)^{n-1}$  by taking the eigenvalues of  $X^i$  to be large and distinct. Classically, there are many other possibilities. It was then shown that for any direction in this vacuum, the electric charge would not be screened, and there would be an energy barrier to making  $X^i$  large if  $n$  and  $m$  were relatively prime. The mass gap allowed the superpotential to be perturbed without changing the ground state degeneracy. The unique solution that screened the charge spontaneously broke the  $SU(n)$  gauge symmetry causing all fields except the center of mass degrees of freedom to be massive.

In our case the gauge symmetry is  $SU(2n)$  with  $n$  and  $m$  relatively prime, and we are on  $S^1/\mathbf{Z}_2$ . If  $m$  is odd, then  $2n$  and  $m$  are relatively prime, and there is an energy barrier in every direction to making the  $X^i$  large. This energy barrier allows one to perturb the superpotential of the  $N = 4$  supersymmetric Yang-Mills theory in four dimensions (this term reduces unchanged to two dimensions where there are additional terms) written in terms of  $N = 1$  fields,

$$S = \text{Tr}(A[B, C]), \quad (4.15)$$

without changing the ground state degeneracy where  $A$ ,  $B$ , and  $C$  are the three chiral superfields in the adjoint representation of  $U(2n)$ . A vacuum state requires that these fields form a  $2n$  dimensional representation of  $SU(2)$ . There is a unique supersymmetric vacuum state in which the  $SU(2n)$  gauge symmetry is spontaneously broken and the boundary charge screened corresponding to the case that the representation is irreducible. By taking the radius of  $S^1/\mathbf{Z}_2$  to infinity, we recover the situation of Ref. [11], and the bound state does not disappear. If  $m$  is even, we must ask whether screening occurs when we break  $SU(2n)$  to  $U(1) \times (SU(n))^2$  by varying the  $T^5$   $G_\beta$  scalars. Here the charges of one of the fundamental  $n$ 's of  $SU(n)$  with respect to  $G_\beta$  will all be  $+1$ , and the charges of the other  $n$  will all be  $-1$ . Since we know that there are  $(n, \frac{m}{2})$  bound states by the

above argument, there would appear to be no energy barrier in this direction. However, the monodromy exchanges the two bound states leaving a unique bound state with an energy barrier to separation in this direction. We conclude that there is a unique bound state whether  $m$  is odd or even. As noted in Ref. [16] the conclusion should not change when some of the  $X^i$  need to be periodically identified (as they do here because we are dimensionally reducing on  $T^5$  not  $R^5$ ) because the bound state is localized in field space. For the same reason, the conclusion should also be valid away from the singularities where the supersymmetry on the brane is halved, and charged hypermultiplets become massless. Since turning on the  $Sp(n)$  Wilson lines in the unreduced theory is equivalent to giving expectation values to the periodic scalars in the dimensionally reduced theory, there must be an energy barrier to activating these lines in the unreduced theory. We have, therefore, found  $16 \times 8$  of the H-dyon states as in Ref. [7] but one-half the number of Ref. [16] because of the  $\mathbf{Z}_2$  action.

#### 4.2. Singular H-Dyons

Let us try to find the other expected  $16 \times 16$  H-dyon states with  $n, m$  relatively prime. We have seen that in the type I' picture there is generically an energy barrier to turning on the  $W_{\beta_i}^\rho$  Wilson lines on  $T^5$ . However, there are special points in the space of  $W_{\beta_i}^\rho$  where the interaction with  $SO(32)$  Wilson lines cause charged fundamentals of the gauge group to become massless. At these points the generic  $N = 2$  supersymmetry in six dimensions will be halved. If we T-dualize on  $T^5$  at these points, we find that at least one onebrane will touch a threebrane there. To obtain the expected H-dyon states, we expect that in general cases there will be no supersymmetric ground states unless all the  $W_{\beta_i}^\rho$  are adjusted so that an extra  $2n$  charged hypermultiplets (in the six dimensional sense of hypermultiplet) become massless. In the case that  $n$  and  $m$  are relatively prime, we require that the  $W_{\beta_i}^\rho$  are such that  $\beta_i = \beta^0$ , all  $i$  where  $\beta^0$  is determined by a  $U(1)$  Wilson line of  $SO(32)$ . Consider  $m$  odd and  $(n, m)$  relatively prime. Fixing the  $W_{\beta_i}^\rho$ , we know that every direction varying the  $W_{\alpha_i}$  lines in  $S^1/\mathbf{Z}_2$  or the  $R^3$  scalars contains an unbroken  $U(1)$  that is charged with respect to every component of the charge on the boundary. Unless all of the nonabelian flux is screened, there is an energy barrier in some of these directions. Suppose we could separate the system into subsystems by moving along directions where the nonabelian flux is screened. The resulting vacuum states could not be supersymmetric because there would be an energy barrier in all the other directions which is not permitted by the  $N = 4$  supersymmetry in two dimensions [11]. If  $m$  is even the above possible

vacuum state is split into two; but because of the  $\mathbf{Z}_2$  monodromy, we also find an energy barrier in all other directions for this case. Thus, we require  $2n$  charged states to become massless so that all of the nonabelian flux can be screened.

We now address the question of why only singularities with all the  $W_{\beta_i}^\rho$  having  $\beta_i = \beta^0$  should give supersymmetric ground states in the relatively prime case. From the above discussion it is clear that there will be an energy cost to varying any of the  $W_{\beta_i}^\rho$  away from the singularity because some of the  $2n$  hypermultiplets will gain a mass. However, we might imagine that there will be another minimum of the potential with several subsystems  $(2n^s, m^s)$  with  $n^s, m^s$  relatively prime such that  $2n = 2 \sum_s n^s$  charged hypermultiplets become massless at this singularity. This cannot occur in the relatively prime case for the following reason. Not all of these subsystems will be identical so that the permutation symmetry will be violated, and we require that a supersymmetric ground state be invariant under the Weyl symmetry. In the case that  $n$  and  $m$  are not relatively prime, there can be such subsystems which respect the permutation symmetry.

There are sixteen singularities that meet the above requirements for  $n, m$  relatively prime. They correspond to adjusting the  $W_{\beta_i}^\rho$  Wilson lines to interact with each of the sixteen  $U(1)$ 's of  $SO(32)$  such that in T-dualizing on  $T^5$  we find  $n$  onebranes touching one of the sixteen threebranes at each singularity. Now we need to argue that each of these singularities gives sixteen bound states. At the singularity the gauge symmetry is  $U(1) \times U(1) \times (SU(n))^2$  where the two  $U(1)$ 's are  $G_\alpha$  and  $G_\beta$ , and the monodromy exchanges the two  $SU(n)$ 's. In addition to the adjoint scalars we have an  $(n, 1) + (1, n)$  of  $(SU(n))^2$  corresponding to an  $N = 2$  hypermultiplet in four dimensions. Since all of the boundary charge can be screened by the charged fundamentals, a vacuum solution requires breaking the gauge symmetry to  $U(1)^{2n}$  by giving expectation values to the adjoint scalars parametrizing  $R^3$  and the  $W_{\alpha_i}$  Wilson lines in the  $S^1$ . These fields do not interact with the charged hypermultiplets. Remember that we can only give expectation values to the neutral scalars in the diagonal  $SU(n)$  of  $SU(n)^2$  because of the monodromy. Including the center of mass scalars, we see that the nonabelian gauge symmetry can only be broken by assigning *distinct* expectation values to the  $n$  neutral ‘‘hypermultiplets’’. By hypermultiplets I mean the four scalars parametrizing  $R^3 \times S^1$ .

We now have to argue that each such configuration of hypermultiplets gives a unique supersymmetric ground state modulo a  $U(n)$  gauge transformation. (Any diagonal  $U(n)$  transformation will preserve the eigenvalues of the adjoint scalars.) Our argument will be similar to that of Ref. [12]. In four dimensions, the  $N = 2$  supersymmetric theory has the

following coupling between the the neutral chiral multiplets of the  $U(1)$  vector multiplets parametrizing two Wilson lines on  $T^5$  and the charged chiral multiplets composing the charged hypermultiplets:

$$S = \sum_{i=1}^n (\bar{\Lambda}_i^1 (\phi_{\beta_i} - \phi_{\beta^0}) \Lambda_i^1 - \bar{\Lambda}_i^2 (\phi_{\beta_i} - \phi_{\beta^0}) \Lambda_i^2) \quad (4.16)$$

where  $\phi_{\beta^0}$  is the value at the singularity. The  $\phi_{\alpha_i}$ 's do not contribute because of the monodromy. We seem to be singling out two directions in  $T^5/\mathbf{Z}_2$  (the space of  $W_{\beta_i}^\rho$  with  $i$  fixed), but the  $SL(5)$  symmetry of  $T^5/\mathbf{Z}_2$  allows us to always rotate the coupling to this form. This coupling should reduce unchanged to two dimensions. Again, the neutral hypermultiplets are uncoupled to these charged fields. There is an energy barrier to perturbing  $\phi_{\beta_i}$  because the charged fields gain a mass and there is an unbroken  $U(1)$  flux that is unscreened as in our previous discussion. Thus,  $S$  can be perturbed by adding some other terms [12], and taking into account the monodromy one finds a unique solution, modulo a  $U(1)^n$  transformation, that breaks the  $U(1)$ 's causing all fields except the neutral hypermultiplets to be massive and screens the electric flux. Thus, the space of solutions is determined by the  $n$  neutral hypermultiplets modulo a gauge transformation. By finding this solution that screens the flux, we have effectively reduced the system to the  $N = 4$  supersymmetric quantum mechanics on the moduli space of hypermultiplets.

Let us return to the heterotic string picture. The hypermultiplets have an interpretation as the location of small instantons in  $R^3 \times S^1$ . These instantons have electric charge determined by the momentum on  $S^1$ . The moduli space of hypermultiplets is, thus, equivalent to the desingularization of  $S^n(R^3 \times S^1)$  ( $S^n$  is the symmetric product of  $n$  elements). It is argued in Ref. [13] that this moduli space is diffeomorphic to that of the BPS  $n$ -monopole space. Since noncompact spaces generally have a choice of desingularizations, and we would not want to treat this space as an orbifold in string theory (an approach which would give the wrong answer), we assume that the Hilbert scheme approach of Ref. [13] is the correct one. Allowing for the action of the charge generator on this space, we obtain the moduli space of BPS  $(n, m)$ -dyons where we have assumed that whether the charge is from winding or momentum does not affect the moduli space. The center of mass can be factored out giving the usual sixteen states. At this point it is natural to conjecture that  $n$  small instantons with momentum  $m$  ( $n, m$  relatively prime) at such a singularity where the nonperturbative gauge symmetry is eaten are the same as a BPS  $(n, m)$ -dyon. Thus, assuming the conjecture of Ref. [5] is correct, we get the extra

$16 \times 16$  states from the singularities, raising the total to  $16 \times 24$  as expected. Note that it is claimed in Ref. [17] that the conjecture of Ref. [5] has been proven, and evidence for the conjecture has also been presented in Ref. [18].

Let us try to make our conjecture more plausible. As in our previous discussion, at the singularity we will have a common  $U(1)$  shared by  $SO(32)$  and the overall  $Sp(1)$  of the center of mass. There will be no nontrivial dependence on  $T^6$  except for the momentum on the  $S^1$  (Assume that we are in the type I picture.) All nonperturbative gauge symmetry is broken so we expect a nonsingular solution. The mass of the resulting state is related to the expectation values of the overall Wilson lines  $W_\beta^\rho$ . Half the supersymmetry on the D5 brane is broken at the singularity with the appearance of a BPS dyon. Finally, the moduli space of the small instantons at the singularity agrees with that of the BPS dyon.

## 5. Conclusions

We have located the  $16 \times 24$   $(n, m)$  H-dyon states with  $n, m$  relatively prime expected by S-duality of the heterotic string. Our results are valid at all points of moduli space where the perturbative gauge symmetry is abelian. Enhanced perturbative gauge symmetry should not change this picture for the  $16 \times 8$  states at generic points of the moduli space, but the  $16 \times 16$  states can only come from small instantons. We have conjectured and found evidence that at singularities where the nonperturbative gauge symmetry is completely broken, the H-dyon bound state should be equivalent to a BPS dyon bound state.

The elementary heterotic states on  $T^4 \times S_5^1 \times S_6^1$  with  $N_L = 1$  that have momentum on  $S_6^1$  correspond to Type IIA states on  $K3 \times S_5^1 \times S_6^1$  with momentum on  $S_6^1$ . The heterotic states that are magnetically charged with respect to  $B_{\mu 6}$  ( $B$  is the antisymmetric tensor) turn into winding states on  $S_5^1$  that are electrically charged with respect to  $B_{\mu 5}$ . The mass of the  $(n, m)$  states in Type IIA is

$$M^2 \sim \frac{m^2}{R_6^2} + \frac{n^2 \times R_5^2}{\alpha'^2} \quad (5.1)$$

so that if  $(n, m)$  are relatively prime, these states are stable. Either by reducing the low energy IIA supergravity on  $K3 \times T^2$  or by calculating directly the IIA theory on a K3 orbifold, one obtains  $24 \times 16$  states with oscillators in their ground states. Thus, the total number of H-dyon states in the Type IIA picture with  $(n, m)$  relatively prime is  $24 \times 16$ , and our results provide further evidence in favor of the string-string duality conjectures in

six dimensions. All of our arguments have required  $n$  and  $m$  to be relatively prime. In other cases, there are marginally bound states [19] that produce degeneracies that are not simply the BPS result multiplied by 24, and we do not expect our conjecture to be valid in these cases.

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